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On Crossing Event Formulas in Critical Two-Dimensional Percolation

Robert S. Maier[1]

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Several formulas for crossing functions arising in the continuum limit of critical two-dimensional percolation models are studied. These include Watts's formula for the horizontal-vertical crossing probability and Cardy's new formula for the expected number of crossing clusters. It is shown that for lattices where conformal invariance holds, they simplify when the spatial domain is taken to be the interior of an equilateral triangle. The two crossing functions can be expressed in terms of an equianharmonic elliptic function with a triangular rotational symmetry. This suggests that rigorous proofs of Watts's formula and Cardy's new formula will be easiest to construct if the underlying lattice is triangular. The simplification in a triangular domain of Schramm's "bulk Cardy's formula" is also studied.

KEY WORDS: Critical percolation; conformal invariance; crossing functions; Watts's formula; special functions.

1. INTRODUCTION

The critical behavior of percolation is not fully understood, either rigorously or formally. As the percolation threshold is approached, connected clusters occur with high probability on ever larger length scales. It has been conjectured that in any dimension, there is a universal scaling limit of isotropic short-range percolation models, defined over the continuum and independent of the details of the model, such as the lattice and percolation type (site or bond).[(2)] This continuum theory would capture the connectivity of typical configurations of the underlying discrete model, at or near criticality.

[1] Departments of Mathematics and Physics, University of Arizona, Tucson, Arizona 85721; e-mail: rsm@math.arizona.edu

Conformal field theory makes predictions for the crossing probabilities of the continuum limit of critical two-dimensional percolation, and especially, predicts that they are conformally invariant.[9] For example, if the continuum theory is confined to a spatial domain Ω plus boundary $\partial\Omega$, and γ_1, γ_2 are disjoint pieces of $\partial\Omega$, the probability of the event that γ_1, γ_2 are connected by a percolation cluster is predicted to be invariant under transformations that are conformal on Ω (though not necessarily on $\partial\Omega$). The probabilities of more complicated crossing events, involving more than two pieces of $\partial\Omega$, are also predicted to be invariant.

In the first applications of conformal field theory to percolation, Ω was taken to be a rectangle, with aspect ratio $r \stackrel{\text{def}}{=} \text{width}/\text{height}$. In this geometry, Cardy[7] derived a formula for the crossing function $\Pi_h(r)$, the probability that the two vertical sides are horizontally connected. His formula takes on a simpler form if the rectangle is conformally mapped onto the upper half plane $\mathbb{H} \subset \mathbb{C}$, and its boundary to $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$. The vertical sides are mapped to disjoint line segments on the real axis, one of which can be taken without loss of generality to be semi-infinite. They are usually taken to be $[0, z]$ and $[1, \infty]$, where $z \in (0, 1)$, with $z = 0, 1/2, 1$ corresponding to $r = \infty, 1, 0$. We write $\mathfrak{P}_h(z) \stackrel{\text{def}}{=} \Pi_h(r(z))$. If the underlying discrete model is bond percolation on a square lattice, duality suggests $\Pi_h(1/r) = 1 - \Pi_h(r)$, i.e., $\mathfrak{P}_h(1-z) = 1 - \mathfrak{P}_h(z)$, so that $\Pi_h(1) = \mathfrak{P}_h(1/2)$, the probability that two opposite sides of a large square are connected by a critical percolation cluster, should equal $1/2$.

On the numerical side, crossing events and their conformal invariance were extensively investigated by Langlands and collaborators,[13, 14] and it was verified that Cardy's formula is valid for discrete percolation models on a rectangular square lattice of size $L \times L'$, with $r = L/L'$, in the limit $L, L' \to \infty$. They also investigated $\Pi_{hv}(r)$, the probability that all four sides of the rectangle are connected. Watts[23] derived a formula for the equivalent function $\mathfrak{P}_{hv}(z) \stackrel{\text{def}}{=} \Pi_{hv}(r(z))$ from conformal field theory by making additional assumptions, and his formula agrees well with the data of Langlands et al. It should be noted that $\mathfrak{P}_{hv} \leqslant \mathfrak{P}_h$, and that by symmetry, $\Pi_{hv}(1/r) = \Pi_{hv}(r)$, i.e., $\mathfrak{P}_{hv}(1-z) = \mathfrak{P}_{hv}(z)$. Also, it is clear that $\Pi_{hv}(r)/\Pi_h(r) \to 1$ as $r \to \infty$, i.e., $\mathfrak{P}_{hv}(z)/\mathfrak{P}_h(z) \to 1$ as $z \to 0$.

Cardy[8, 9] later derived a formula for the expected number of percolation clusters that cross between γ_1, γ_2 (the left and right-hand sides of a rectangle, or the two corresponding line segments in $\partial\mathbb{H}$). This may be viewed as a function $N_h(r)$, or equivalently a function $\mathfrak{N}_h(z) \stackrel{\text{def}}{=} N_h(r(z))$; necessarily, $\mathfrak{N}_h \geqslant \mathfrak{P}_h$.

So in all, three crossing formulas for critical percolation have been derived from conformal field theory: formulas for $\mathfrak{P}_h(z)$, $\mathfrak{P}_{hv}(z)$, and $\mathfrak{N}_h(z)$, which yield formulas for $\Pi_h(r)$, $\Pi_{hv}(r)$, and $N_h(r)$. Recently, Smirnov[22] provided the first rigorous proof of any of these, namely Cardy's formula for $\mathfrak{P}_h(z)$. He showed that critical site percolation on the triangular lattice has a conformally invariant scaling limit, and that discrete percolation cluster boundaries converge to a stochastic Loewner evolution process. These facts followed from his proof that in a general domain Ω, a conformally transformed Cardy's formula is valid. This includes a version due to L. Carleson (unpublished, but see ref. 9, Section 7.2, and ref. 24, Prediction 7). Suppose $\mathbb{H}$ is mapped conformally onto an equilateral triangle $\triangle ABC$ in the complex plane, and that the map extends to the boundary, with $[-\infty, 0]$, $[0, 1]$, $[1, \infty]$ being mapped to the edges AB, BC, CA. So $w \in BC$ and $z \in (0, 1)$ will correspond. Carleson noticed that $\tilde{\mathfrak{P}}_h(w) \stackrel{\text{def}}{=} \mathfrak{P}_h(z(w))$ simply equals $\overline{Bw}/\overline{BC}$. That is, Cardy's formula is formally equivalent to the statement that that the line segment Bw is connected by a critical percolation cluster to the opposite side of the triangle, CA, with probability equal to the fraction of the side BC occupied by Bw. Equivalently, the real-valued function $\tilde{\mathfrak{P}}_h(w)$ is the restriction to one side of the triangle of a particularly simple analytic function of w: a *linear* function.

The triangular symmetry of Carleson's restatement made possible Smirnov's proof, which is specific to a triangular lattice. Smirnov notes, "It seems that $2\pi/3$ rotational symmetry enters in our paper not because of the specific lattice we consider, but rather [because it] manifests some symmetry laws characteristic to (continuum) percolation." Whether his proof extends to other lattices is unclear.

In this paper we study whether the predicted formulas for the functions $\mathfrak{P}_{hv}(z)$ and $\mathfrak{N}_h(z)$, like Cardy's formula for $\mathfrak{P}_h(z)$, simplify when the spatial domain Ω is taken to be an equilateral triangle, rather than a rectangle or the upper half plane $\mathbb{H}$. We show that they do. In particular, we show that the four-way crossing function $\tilde{\mathfrak{P}}_{hv}(w) \stackrel{\text{def}}{=} \mathfrak{P}_{hv}(z(w))$ predicted for the equilateral triangle has its second derivative $\tilde{\mathfrak{P}}''_{hv}(w)$ equal to a familiar elliptic function: an equianharmonic Weierstrass $\wp$-function, where "equianharmonic" signifies that the period lattice of the $\wp$-function is triangular, with a $\pi/3$ rotational symmetry. (See ref. 1, Section 18.13.) This contrasts with the linear function $\tilde{\mathfrak{P}}_h(w)$, the second derivative of which is zero. Our new representation for $\tilde{\mathfrak{P}}_{hv}(w)$, i.e., for $\mathfrak{P}_{hv}(z)$ or $\Pi_{hv}(r)$, immediately yields a simple closed-form expression for $\mathfrak{P}_{hv}(1/2) = \Pi_{hv}(1)$, the probability that all four sides of a large square are connected by a critical percolation cluster; namely, $1/4 + (\sqrt{3}/4\pi)(3\log 3 - 4\log 2) \approx 0.322$. It would be difficult though not impossible to derive this expression directly from Watts's formula.

We show that Cardy's recent formula for $\mathfrak{N}_h(z)$, like Watts's formula for $\mathfrak{P}_{hv}(z)$, simplifies in an equilateral triangle. $\tilde{\mathfrak{N}}''_h(w)$, the second derivative of $\tilde{\mathfrak{N}}_h(w) \stackrel{\text{def}}{=} \mathfrak{N}(z(w))$, can also be expressed in terms of the equianharmonic $\wp$-function. In fact, we derive a curious identity relating the three crossing functions $\tilde{\mathfrak{P}}_h$, $\tilde{\mathfrak{P}}_{hv}$, and $\tilde{\mathfrak{N}}_h$, or equivalently $\mathfrak{P}_h$, $\mathfrak{P}_{hv}$, and $\mathfrak{N}_h$; namely, that $2\mathfrak{N}_h(z)-\mathfrak{P}_h(z)-\mathfrak{P}_{hv}(z)$ must equal $(\sqrt{3}/2\pi)\log(1/(1-z))$. Setting $z=1/2$ yields that $\mathfrak{N}_h(1/2)$, i.e., $N_h(1)$, the expected number of critical percolation clusters crossing between two opposite sides of a large square, should equal $3/8+(\sqrt{3}/8\pi)(3\log 3-2\log 2)\approx 0.507$. In a final study of a crossing formula, we treat a fourth Cardy-type formula proved rigorously by Schramm[19] for triangular-lattice site percolation, which is valid "in the bulk" and does not follow from boundary conformal field theory. We show it has a simple restatement in a suitable triangular domain.

Our successful simplification of the crossing event formulas indicates that an equilateral triangular domain is a good "fit" to the continuum limit of critical percolation. It also suggests that rigorous proofs of Watts's formula and Cardy's new formula will be easiest to construct if the underlying lattice is triangular. Our restatements contrast with those of Ziff[28, 29] and Kleban and Zagier,[11, 12] which also involve higher transcendental functions. They focused on the continuum limit of percolation in a rectangle, and especially on the derivatives $\Pi'_h(r)$ and $\Pi'_{hv}(r)$ of the rectangular crossing functions. In our notation, Ziff showed that $\Pi'_h(r)$ is proportional to $[\theta'_1(0, q=e^{-\pi r})]^{4/3}$, where $\theta_1(\cdot, q)$ is the first Jacobi theta function. Also, Kleban and Zagier showed that Π'_h, Π'_{hv}, considered jointly, have interesting modular transformation properties, and that these properties characterize $\Pi'_{hv}(r)$. The relation between our results and theirs is not yet clear.

Section 2 presents each crossing event formula in a standard form. Section 3 covers conformal mapping concepts, including the equianharmonic $\wp$-function. The simplified versions of the crossing formulas that apply in triangular domains are derived in Section 4. An appendix reviews some basic mathematical facts.

2. CROSSING EVENT FORMULAS

The four crossing formulas for continuum percolation on the closed upper half plane $\overline{\mathbb{H}}$ involve hypergeometric functions, both Gauss's ${}_2F_1$ and Clausen's ${}_3F_2$. They can be stated in a standardized, P-symbol form. (For hypergeometric functions and P-symbols, see the appendix.) For the first three formulas, $\partial\mathbb{H}$ is divided into $[-\infty, 0]$, $[0, z]$, $[z, 1]$, and $[1, \infty]$. A connection between $[0, z]$ and $[1, \infty]$ corresponds to a horizontal crossing on the original rectangle, and one between $[-\infty, 0]$ and $[z, 1]$ to a vertical crossing. The probability of a horizontal connection is $\mathfrak{P}_h(z)$, with $\mathfrak{N}_h(z)$ the expected number of such connections. All four segments are connected with probability $\mathfrak{P}_{hv}(z)$. Let $\mathfrak{P}_{h\bar{v}} \stackrel{\text{def}}{=} \mathfrak{P}_h - \mathfrak{P}_{hv}$, the probability of there being a horizontal connection that is not also a vertical one.

Formula 2.1 (Cardy [7]). The function $\mathfrak{P}_h(z)$ equals

$$\frac{3\,\Gamma(2/3)}{\Gamma(1/3)^2}\, z^{1/3}{}_2F_1\left(\begin{matrix}1/3, 2/3\\ 4/3\end{matrix}\,\middle|\, z\right) \propto P\left\{\begin{array}{ccc|c} 0 & 1 & \infty & z \\ \hline 0 & 0 & 0 & \\ \boxed{1/3} & 1/3 & 1/3 & \end{array}\right\}. \tag{1}$$

Formula 2.2 (Watts [23]). The function $\mathfrak{P}_{h\bar{v}}(z)$ equals

$$\frac{\sqrt{3}}{2\pi}\, z\, {}_3F_2\left(\begin{matrix}1, 1, 4/3\\ 2, 5/3\end{matrix}\,\middle|\, z\right) \propto P\left\{\begin{array}{ccc|c} 0 & 1 & \infty & z \\ \hline 0 & 0 & 0 & \\ 1/3 & 1/3 & 1/3 & \\ \boxed{1} & 1 & 0 & \end{array}\right\}. \tag{2}$$

Formula 2.3 (Cardy [8, 9]). The function $\mathfrak{P}_{h\bar{v}}(z)$ equals

$$\frac{1}{2} - \frac{\sqrt{3}}{4\pi}\left[\log(1-z) + (1-z)\, {}_3F_2\left(\begin{matrix}1, 1, 4/3\\ 2, 5/3\end{matrix}\,\middle|\, 1-z\right)\right]. \tag{3}$$

Proof. This version of the formula for $\mathfrak{N}_h(z)$ is not well known. The version deduced by Cardy from boundary conformal field theory was

$$\frac{1}{2} - \frac{\sqrt{3}}{4\pi}\left[\log(1-z) + 2\sum_{m=1}^{\infty} \frac{(1/3)_m}{(2/3)_m}\frac{(1-z)^m}{m}\right]. \tag{4}$$

Formula 2.3 follows from the series representation (A.1), if the $1/m$ factor in the summand is written as $(1)_{m-1}\,(1)_{m-1}/(2)_{m-1}\,(m-1)!$. ▮

Corollary 2.3.1. $2\mathfrak{N}_h(z) - \mathfrak{P}_h(z) - \mathfrak{P}_{hv}(z) = (\sqrt{3}/2\pi)\log(1/(1-z))$.

Proof. This follows by combining Formulas 2.1–2.3, with the aid of the symmetry relations $\mathfrak{P}_h(1-z) = 1 - \mathfrak{P}_h(z)$ and $\mathfrak{P}_{hv}(1-z) = \mathfrak{P}_{hv}(z)$. ▮

Corollary 2.3.1 ties $\mathfrak{N}_h$ to $\mathfrak{P}_h$ and $\mathfrak{P}_{hv}$ in quite a strong way. To explain how, we must sketch the heuristic origins of Formula 2.2 in boundary conformal field theory. Watts was led to Formula 2.2 by considering ODEs satisfied by correlation functions of boundary operators. His candidate for an ODE satisfied by $\mathfrak{P}_{hv}$ was the fifth-order Fuchsian equation

$$\left\{ \frac{d^3}{dz^3} [z(z-1)]^{4/3} \frac{d}{dz} [z(z-1)]^{2/3} \frac{d}{dz} \right\} F = 0, \tag{5}$$

which has P-symbol

$$P \left\{ \begin{array}{ccc|c} 0 & 1 & \infty & z \\ \hline 0 & 0 & 0 & \\ 1/3 & 1/3 & 1/3 & \\ 0 & 0 & 0 & \\ 1 & 1 & 1 & \\ 2 & 2 & 2 & \end{array} \right\}, \tag{6}$$

but is not of hypergeometric type. Due to a factorization of the differential operator on the left-hand side of this equation,[11, 12] its solution space properly contains the solution space of the third-order equation

$$\left\{ \frac{d}{dz} [z(z-1)]^{1/3} \frac{d}{dz} [z(z-1)]^{2/3} \frac{d}{dz} \right\} F = 0, \tag{7}$$

which has P-symbol

$$P \left\{ \begin{array}{ccc|c} 0 & 1 & \infty & z \\ \hline 0 & 0 & 0 & \\ 1/3 & 1/3 & 1/3 & \\ 1 & 1 & 0 & \end{array} \right\}, \tag{8}$$

and is of hypergeometric type. Furthermore, the solution space of (7) properly contains the solution space of the second-order equation with P-symbol

$$P \left\{ \begin{array}{ccc|c} 0 & 1 & \infty & z \\ \hline 0 & 0 & 0 & \\ 1/3 & 1/3 & 1/3 & \end{array} \right\}, \tag{9}$$

including the function $\mathfrak{P}_h$ of Formula 2.1. Watts noticed that the solution space of (7) includes a one-dimensional subspace of functions equal to zero at $z=0$ and invariant under $z\mapsto 1-z$, which are criteria for the function $\mathfrak{P}_{hv}$. For this reason, he expected $\mathfrak{P}_{hv}$, and $\mathfrak{P}_{h\bar{v}}=\mathfrak{P}_h-\mathfrak{P}_{hv}$ too, to be solutions of (7), as well as of (5). This insight led to Formula 1, which incorporates the P-symbol (8).

Kleban and Zagier[12] noticed that the five-dimensional solution space of (5) is spanned by the solution space of (7), to which $\mathfrak{P}_{hv}$ and $\mathfrak{P}_h$ belong, and the functions $\log z$ and $\log(1-z)$. But by Corollary 2.3.1, the function $\mathfrak{N}_h$ is a linear combination of $\mathfrak{P}_{hv}$, $\mathfrak{P}_h$, and $\log(1-z)$, implying the following mysterious fact.

Corollary 2.3.2. The function $\mathfrak{N}_h(z)$, like the functions $\mathfrak{P}_{hv}(z)$ and $\mathfrak{P}_h(z)$, is a solution of Watts's fifth-order differential equation, Eq. (5).

For the fourth crossing formula, $\partial\mathbb{H}$ is divided into $[-\infty,z]$ and $[z,\infty]$, with $z\in\mathbb{R}$ unrestricted. A special boundary condition is imposed: on the underlying discrete lattice, percolation along the line segment $[-\infty,z]$ is allowed by fiat. Let a distinguished point in $\mathbb{H}$ be chosen; without loss of generality, let it be $i=\sqrt{-1}$. Then the function $\mathfrak{P}_{\mathrm{surr}}(z)$ is defined to be the probability that i is surrounded by the percolation hull of $[-\infty,z]$, i.e., the outermost boundary of the percolation cluster in $\mathbb{H}$ containing ("growing from") $[-\infty,z]$. One expects $\mathfrak{P}_{\mathrm{surr}}(-z)=1-\mathfrak{P}_{\mathrm{surr}}(z)$, i.e., that $\mathfrak{P}_{\mathrm{surr}}(z)-1/2$ is odd in z.

Formula 2.4 (Schramm [19]). The function $\mathfrak{P}_{\mathrm{surr}}(z)$ equals $1/2$ plus

$$\frac{\Gamma(2/3)}{\sqrt{\pi}\,\Gamma(1/6)}\,z\,{}_2F_1\left(\begin{matrix}1/2,\,2/3\\3/2\end{matrix}\,\middle|\,-z^2\right)\propto P\left\{\begin{array}{ccc|c}0&1&\infty&-z^2\\\hline 0&0&0&\\\boxed{1/2}&1/3&1/6&\end{array}\right\}.$$

In the P-symbol expression on the right, $z<0$ and $z>0$ correspond to different branches, which are negatives of each other.

3. CONFORMAL MAPS AND FUCHSIAN ODEs

The upper half plane $\mathbb{H}$ and any triangle $\triangle ABC$ without boundary are homeomorphic as complex manifolds. In fact, by the Poincaré–Koebe uniformization theorem, the complex plane $\mathbb{C}$, the Riemann sphere $\mathbb{CP}^1 \stackrel{\text{def}}{=} \mathbb{C} \cup \{\infty\}$, and $\mathbb{H}$ are the only simply connected one-dimensional complex manifolds, up to conformal equivalence.[6] To transfer the Fuchsian ODEs satisfied by the crossing functions from $\mathbb{H}$ to $\triangle ABC$, an explicit expression for the conformal map $w = s(z)$, i.e., $s\colon \mathbb{H} \to \triangle ABC$, is useful. s is a Schwarz triangle function, defined by the Schwarz–Christoffel formula.[18] It extends to a map from the closure of $\mathbb{H}$ in $\mathbb{CP}^1$, i.e., $\mathbb{H} \cup \mathbb{R} \cup \{\infty\}$, to the triangle with boundary. The conditions $s(0) = B$, $s(1) = C$, $s(\infty) = A$ uniquely determine s.

The inverse Schwarz function $S\colon \triangle ABC \to \mathbb{H}$ often has a deeper significance than s does, as the equilateral triangle case illustrates. Equilateral triangles tile the plane, and the inverse function extends to a function $S\colon \mathbb{C} \to \mathbb{CP}^1$ that maps alternating triangles in a checkerboard fashion to the upper and lower half planes. In modern treatments, this extended map is classed as one of a handful of "universal" branched covers of $\mathbb{CP}^1$ by $\mathbb{C}$, $\mathbb{CP}^1$, or $\mathbb{H}$ (see ref. 20, Section 6.4).

Since the triangular lattice is doubly periodic, one expects that S is an elliptic function, and can equally well be viewed as a function on a complex elliptic curve $\mathbb{C}/\mathscr{L}$, where $\mathscr{L} \stackrel{\text{def}}{=} 2\omega\mathbb{Z} + 2\omega'\mathbb{Z}$ is an appropriate lattice of periods (the factors of 2 are traditional). This is correct, as we briefly sketch; for details, see Abramowitz and Stegun,[1] Section 18.13, and Sansone and Gerretsen,[18] Sections 14.2 and 14.3. The canonical elliptic function on $\mathbb{C}$ is the Weierstrass function $\wp(w; g_2, g_3)$, defined as the solution of $(\wp')^2 = f(\wp) \stackrel{\text{def}}{=} 4\wp^3 - g_2\wp - g_3$ with a unit-strength double pole at $w = 0$. The parameters $g_2, g_3 \in \mathbb{C}$, both of which cannot be zero, are related in a nontrivial way to the fundamental half-periods ω, ω' of $\wp$. The *equianharmonic* case, which has special symmetries, is the case when $g_2 = 0$. Due to the homogeneity relation $\wp(w; g_2, g_3) = t^2\wp(tw; t^{-4}g_2, t^{-6}g_3)$, in the equianharmonic case all nonzero g_3 are equivalent, so henceforth $g_3 \in \mathbb{R}\setminus\{0\}$, in particular $g_3 = 1$, will be taken. In this case the fundamental half-periods ω, ω' can be chosen to be a complex conjugate pair with $\Re\omega > 0$, $\Im\omega < 0$. If the basic real half-period $\omega + \omega' > 0$ is denoted ω_2, then ω, ω' will equal $(\frac{1}{2} \mp \frac{\sqrt{-3}}{2})\,\omega_2$. So the period lattice $\mathscr{L}$ will be a triangular lattice, with a $\pi/3$ rotational symmetry about the origin. Explicitly, $\omega_2 = \Gamma(1/3)^3/4\pi \approx 1.530$.

The elliptic curve $\mathbb{C}/\mathscr{L}$ is homeomorphic to a torus and can be viewed as the parallelogram with vertices 0, 2ω, $2\omega_2$, $2\omega'$, equipped with periodic boundary conditions. The $\wp$-function maps this parallelogram doubly onto $\mathbb{CP}^1$. Also, $\wp'$ maps it triply onto $\mathbb{CP}^1$. It turns out that in the equianharmonic case, the torus, i.e., this period parallelogram, can be subdivided into six equilateral triangles, each mapped by $\wp'$ with unit multiplicity onto the left or right half plane. (See ref. 1, Fig. 18.11, which is unfortunately not quite to scale.) Due to this, the equilateral inverse Schwarz function S can be chosen to be essentially $\wp'$. The map

$$z = S(w) \stackrel{\text{def}}{=} 1/2 + \wp'(w)/2i \tag{10}$$

will take the equilateral triangle $\triangle ABC \stackrel{\text{def}}{=} \triangle 0\overline{W_0}W_0$ in w-space to the upper half plane in z-space, where $W_0 \stackrel{\text{def}}{=} (1+\frac{\sqrt{-3}}{3})\,\omega_2$. It will take the boundary of $\triangle ABC$ to $\mathbb{R} \cup \{\infty\}$, and the vertices $w = A, B, C$ respectively to $z = \infty, 0, 1$. As a map from $\mathbb{C}$ to $\mathbb{CP}^1$, it will take alternating triangles to the upper and lower half planes. Each of these equilateral triangles, which tile the plane, has one vertex in each of the congruence classes $A+\mathscr{L}, B+\mathscr{L}, C+\mathscr{L}$, i.e., in each of the sets $S^{-1}(\infty) = \mathscr{L}, S^{-1}(0), S^{-1}(1)$. These classes will be denoted $[A]$, $[B]$, $[C]$.

The ODEs on $\mathbb{CP}^1 \supset \mathbb{H}$ that are satisfied by the functions of Formulas 2.1–2.3 can be pulled back to ODEs on $\mathbb{C}$ via the extended map S: $\mathbb{C} \to \mathbb{CP}^1$. The important thing to note when performing the pullback is that S is a *branched* cover of $\mathbb{CP}^1$ by $\mathbb{C}$, the critical points of which are the points in $[A]$, $[B]$, and $[C]$. At any $w_0 \in \mathbb{C}$, necessarily $S(w) \sim S(w_0) + \text{const} \times (w-w_0)^p$ to leading order, where p is the multiplicity with which w_0 is mapped to $S(w_0)$. (If $S(w_0) = \infty$, the right-hand side must be replaced by $\text{const} \times (w-w_0)^{-p}$.) Each critical point has $p = 3$.

The pulled-back ODEs are tightly constrained by the following lemma, which specifies how P-symbols are pulled back. It is proved by considering the local (Frobenius) solutions at each point.

Lemma 3.1. Let R: $M \to \mathbb{CP}^1$ be a holomorphic map of one-dimensional complex manifolds. Consider $Lu = 0$, an nth-order Fuchsian ODE on $\mathbb{CP}^1$. It can be pulled back via R to a Fuchsian ODE $\tilde{L}\tilde{u} = 0$ on M, in the sense that if $u = u(z)$ satisfies $Lu = 0$ then $\tilde{u} = \tilde{u}(w) \stackrel{\text{def}}{=} u(R(w))$ will satisfy $\tilde{L}\tilde{u} = 0$. The n characteristic exponents of $\tilde{L}$ at each $w_0 \in M$ will equal those of L at $z_0 = R(w_0)$, multiplied by the multiplicity with which w_0 is mapped to z_0.

As an illustration of the use of this lemma in pulling back ODEs, we give a new proof of Whipple's quadratic transformation formula for ${}_3F_2$. (The formula originally appeared in ref. 25, with a combinatorial proof; a simpler combinatorial proof is due to Bailey.[5] For a useful discussion placing the formula in context, see Askey,[4] but note the misprint in Eq. (2.19).) This new proof resembles Riemann's concise P-symbol proof of Kummer's quadratic transformation formulas for ${}_2F_1$, which is summarized in ref. 3, Section 3.9. It relies on the expression (A.4) for the P-symbol associated to ${}_{q+1}F_q$, which is apparently not well known.

Proposition 3.2. Let $a, b, c \in \mathbb{C}$ with neither $a-b+1$ nor $a-c+1$ equal to a nonpositive integer. Then

$$
\begin{aligned}
&{}_3F_2\left(\left.\begin{matrix} a, b, c \\ a-b+1, a-c+1 \end{matrix}\right| w\right) \\
&\quad = (1-w)^{-a}\,{}_3F_2\left(\left.\begin{matrix} a-b-c+1, a/2, (a+1)/2 \\ a-b+1, a-c+1 \end{matrix}\right| \frac{-4w}{(1-w)^2}\right)
\end{aligned}
$$

holds in a neighborhood of $w = 0$.

Remark. The two sides are defined for all w such that $|w| < 1$, resp. for all w such that $|-4w/(1-w)^2| < 1$. So by analytic continuation, equality holds at all points within the loop of the curve $|4w| = |1-w|^2$ surrounding the origin.

Proof. The functions of w on the two sides satisfy third-order Fuchsian ODEs that follow from the $q = 2$ case of (A2), the ODE satisfied by ${}_{q+1}F_q$. The left and right-hand functions are determined by the additional condition that they be analytic at $w = 0$ and equal unity there. It will therefore suffice to prove that the Fuchsian ODEs corresponding to the two sides are the same up to normalization, i.e., have the same solution spaces. A necessary condition for this is that their P-symbols be the same, i.e., by the representation (A.4), that

$$
\begin{aligned}
&P\left\{\begin{array}{ccc|c} 0 & 1 & \infty & w \\ \hline 0 & 0 & a & \\ b-a & 1 & b & \\ c-a & a-2b-2c+2 & c & \end{array}\right\} \\
&\quad = (1-w)^{-a}\, P\left\{\begin{array}{ccc|c} 0 & 1 & \infty & R(w) \\ \hline 0 & 0 & a-b-c+1 & \\ b-a & 1 & a/2 & \\ c-a & 1/2 & (a+1)/2 & \end{array}\right\}, \qquad (11)
\end{aligned}
$$

where $R\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ is defined by $R(w) \stackrel{\text{def}}{=} -4w/(1-w)^2$, or equivalently

$$P\left\{\begin{array}{ccc|c} 0 & 1 & \infty & w \\ \hline 0 & a & 0 & \\ b-a & a+1 & b-a & \\ c-a & 2(a-b-c+1) & c-a & \end{array}\right\} = P\left\{\begin{array}{ccc|c} 0 & 1 & \infty & R(w) \\ \hline 0 & 0 & a-b-c+1 & \\ b-a & 1 & a/2 & \\ c-a & 1/2 & (a+1)/2 & \end{array}\right\}. \tag{12}$$

That is, R must pull back the right-hand P-symbol in (12) to the left-hand one.

The map $w \mapsto z \stackrel{\text{def}}{=} R(w)$ takes $w = 0, 1, \infty$ to $z = 0, \infty, 0$ respectively, and also $w = -1$ to $z = 1$. Its critical points are $w = \pm 1$, each of which has double multiplicity. The exponents in the columns of (12) agree precisely with what Lemma 3.1 states: the exponents of $w = 0$ and $w = \infty$ are the same as those of $z = 0$, and those of $w = 1$ are twice those of $z = \infty$. One might think the left-hand P-symbol would have a fourth critical point, at $w = -1$, with exponents twice those of $z = 1$, i.e., 0, 1, 2. But as noted in the appendix, those exponents are the signature of an ordinary point; so no fourth column is present.

The preceding argument shows why the ${}_3F_2$ parameters of the proposition take the values they do, but it does not quite prove the proposition. As reviewed in the appendix, any Fuchsian ODE on $\mathbb{CP}^1$ that has three singular points and is of third order (i.e., $q = 2$) has $3q+2 = 8$ independent exponent parameters, which are displayed in its P-symbol, and $\binom{q}{2} = 1$ accessory parameter, which is not. For the two ODEs to be the same up to normalization, they must have the same P-symbol, and also the same accessory parameter. The latter is most readily verified by changing variables from $z = R(w)$ to w in the right-hand ODE. An explicit computation, omitted here, shows that the resulting pulled-back ODE is indeed the $q = 2$ case of the ODE (A2), with the parameters of the left-hand side. ∎

What will be used in Section 4 is the following variant of Whipple's quadratic transformation. It seems not to have appeared in the literature.

Proposition 3.3. Let $a, b, c \in \mathbb{C}$ with $(a+b+c)/2$ not equal to a nonpositive integer. Then

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix} a,\, b,\, c \\ 2,\, (a+b+c)/2 \end{matrix}\,\middle|\, w\right) \\
&\quad = (1-w)\,{}_3F_2\left(\begin{matrix} (a+1)/2,\, (b+1)/2,\, (c+1)/2 \\ 2,\, (a+b+c)/2 \end{matrix}\,\middle|\, 4w(1-w)\right)
\end{aligned}
$$

holds in a neighborhood of $w=0$, provided one of a, b, c equals unity.

Remark. The two sides are defined for all w such that $|w|<1$, resp. for all w such that $|4w(1-w)|<1$. So by analytic continuation, equality holds at all points that are both within the circle $|w|=1$ and within the loop of the curve $|4w(1-w)|=1$ surrounding the origin.

Proof. This closely follows that of Proposition 3.2; the details are left to the reader. The only new feature is that equality between the accessory parameters of the ODEs satisfied by the two sides leads to an additional condition on the ${}_3F_2$ parameters, beside the exponent conditions of Lemma 3.1. Changing variables from $z = R(w) \stackrel{\text{def}}{=} 4w(1-w)$ to w in the right-hand ODE pulls it back to the left-hand ODE, plus an extraneous term proportional to $(a-1)(b-1)(c-1)$. Provided one of a, b, c equals unity, this undesired term is absent. ∎

4. TRANSFORMED AND RESTATED FORMULAS

In this section we show how the crossing event formulas, Formulas 2.1–2.4, simplify in appropriately chosen triangular domains. The restated formulas appear in Propositions 4.1–4.4. For all but Schramm's formula, the appropriate triangle is equilateral. The restatements of Cardy's formula and Schramm's formula do not involve special functions. The restatements of Watts's formula and Cardy's new formula do involve elliptic functions, but elliptic functions are significantly more familiar than Clausen's ${}_3F_2$. The restatement of Cardy's formula is of course identical to Carleson's, but the other three are new.

The Fuchsian ODEs on $\mathbb{CP}^1 \supset \mathbb{H}$ of Formulas 2.1–2.3 are pulled back to ODEs on $\mathbb{C} \supset \triangle ABC$ via the inverse Schwarz function $S\colon \mathbb{C} \to \mathbb{CP}^1$. In the normalization of the last section, $z = S(w)$ equals $1/2 + \wp'(w)/2i$, with $\wp$ the equianharmonic $\wp$-function, satisfying $(\wp')^2 = 4\wp^3 - 1$. The equilateral triangle $\triangle ABC$ is $\triangle 0\overline{W_0}W_0$, i.e., $\triangle 0,\, \rho e^{-i\pi/6},\, \rho e^{i\pi/6}$, with side length $\rho \stackrel{\text{def}}{=} |W_0| = 2\omega_2/\sqrt{3}$. As noted, $\omega_2 \stackrel{\text{def}}{=} \Gamma(1/3)^3/4\pi \approx 1.530$ is the basic real half-period of $\wp$.

Since S maps AB, BC, CA to $[-\infty, 0]$, $[0, 1]$, $[1, \infty]$ respectively, the line segment $BC = \overline{W_0}W_0$ is of primary interest. Its midpoint is $\omega_2 = (\overline{W_0}+W_0)/2$, which is mapped to $1/2$. As a necessary preliminary, the behavior of the first few antiderivatives of $\wp'$ along BC will now be described. (See the tables in ref. 1, Section 18.13, where W_0 is denoted "z_0".) Relative to the midpoint, $\wp'$ is an odd function: it equals $-i$ at $\overline{W_0}$ and i at W_0. Its antiderivative $\wp$ is even: it equals zero at $\overline{W_0}$ and W_0, and $4^{-1/3}$ at the midpoint. The negative antiderivative of $\wp$ is the so-called Weierstrass zeta function, plus an arbitrary constant. The shifted negative antiderivative $\zeta - \pi/2\sqrt{3}\,\omega_2$ is odd: it equals $i\pi/6\omega_2$ at $\overline{W_0}$ and $-i\pi/6\omega_2$ at W_0. The antiderivative of ζ equals $\log\sigma$ plus an arbitrary constant, where σ is the Weierstrass sigma function, which equals $e^{\pi/3\sqrt{3}}e^{-i\pi/6}$ at $\overline{W_0}$ and $e^{\pi/3\sqrt{3}}e^{i\pi/6}$ at W_0; and $e^{\pi/4\sqrt{3}}2^{1/3}3^{-1/4}$ at the midpoint ω_2. It is easily checked that

$$-\operatorname{Log}\sigma(w) + \frac{\pi}{2\sqrt{3}\,\omega_2}\,w - \frac{\pi}{6\sqrt{3}}, \tag{13}$$

which is a double antiderivative of $\wp$, is an even function relative to the midpoint: it equals zero at $\overline{W_0}$ and W_0, and $(1/12)(\pi/\sqrt{3} + 3\log 3 - 4\log 2)$ at the midpoint. Log signifies the principal branch of the logarithm function.

Proposition 4.1 (Cardy's Formula, Transformed; cf. Carleson). If conformal invariance holds, Formula 2.1 corresponds on the equilateral triangle $\triangle ABC$ plus boundary to the following. $\tilde{\mathfrak{P}}_h(w)$, the probability that the boundary segments Bw and CA are connected by a percolation cluster, is the restriction to BC of an analytic function that is *linear*. Explicitly, $\tilde{\mathfrak{P}}_h(w) = (w-B)/(C-B)$.

Proof. By Lemma 3.1, the P-symbol of Formula 2 is pulled back via $z = S(w)$ to

$$P\left\{\begin{array}{ccc|c} [A] & [B] & [C] & w \\ \hline 0 & 0 & 0 & \\ 1 & \boxed{1} & 1 & \end{array}\right\}, \tag{14}$$

where $[A], [B], [C]$ are the classes of points on the w-plane that are mapped by S to $z = \infty$, 0, 1 respectively. This is because these points are the critical points of S, and each has triple multiplicity. But a singular point with exponents 0, 1 is effectively an ordinary point. So the pulled-back ODE on $\mathbb{C}$ (in particular, on $\triangle ABC$) has no singular points and should be effectively $(d^2/dw^2)\,\tilde{\mathfrak{P}}_h(w)=0$, as can be verified by an explicit computation. ∎

Proposition 4.2 (Watts's Formula, Transformed). If conformal invariance holds, Formula 2.2 corresponds on the equilateral triangle $\triangle ABC$ plus boundary to the following. $\tilde{\mathfrak{P}}_{hv}(w)$, the probability that all four boundary segments AB, Bw, wC, and CA are connected by a percolation cluster, is the restriction to BC of an analytic function with the property that the difference $\tilde{\mathfrak{P}}_{h\bar{v}}(w)\stackrel{\text{def}}{=}\tilde{\mathfrak{P}}_h(w)-\tilde{\mathfrak{P}}_{hv}(w)$ is proportional to $(w-B)^3$ as $w\to B$, to leading order. Explicitly,

$$\tilde{\mathfrak{P}}_{hv}(w)=-\frac{3\sqrt{3}}{\pi}\operatorname{Log}\sigma(w)+\frac{3}{2}\frac{w}{\omega_2}-\frac{1}{2}, \tag{15}$$

where σ is the equianharmonic Weierstrass sigma function.

Proof. By Lemma 3.1, the P-symbol of Formula 2.1, which partially specifies the ODE satisfied by $\mathfrak{P}_{hv}(z)$ and $\mathfrak{P}_h(z)$, is pulled back via $z=S(w)$ to

$$P\left\{\begin{array}{ccc|c} [A] & [B] & [C] & w \\ \hline 0 & 0 & 0 & \\ 1 & 1 & 1 & \\ 0 & \boxed{3} & 3 & \end{array}\right\}. \tag{16}$$

The condition $\Pi_{hv}(r)/\Pi_h(r)\to 1$ as $r\to\infty$, i.e., $\mathfrak{P}_{hv}(z)/\mathfrak{P}_h(z)\to 1$ as $z\to 0$, implies $\tilde{\mathfrak{P}}_{hv}(w)/\tilde{\mathfrak{P}}_h(w)\to 1$ as $w\to B$. So $\tilde{\mathfrak{P}}_{hv}(w)$ is linear in w as $w\to B$, to leading order. By the P-symbol (16), the first nonzero correction must be cubic.

By changing variables from $z=S(w)$ to w in (7), the third-order ODE satisfied by $\mathfrak{P}_{hv}(z)$ and $\mathfrak{P}_h(z)$, one obtains the striking pulled-back ODE

$$\frac{d}{dw}\left\{[\wp(w)]^{-1}\,\tilde{\mathfrak{P}}_{hv}''(w)\right\}=0 \tag{17}$$

on $\mathbb{C}$. So $\tilde{\mathfrak{P}}_{hv}$ must be proportional to a double antiderivative of $\wp$. The condition $\Pi_{hv}(1/r)=\Pi_{hv}(r)$, i.e., $\mathfrak{P}_{hv}(1-z)=\mathfrak{P}_{hv}(z)$, implies that on the line segment BC, $\tilde{\mathfrak{P}}_{hv}$ must be even around the midpoint $w=\omega_2$. Moreover, the condition $\tilde{\mathfrak{P}}_{hv}(w)/\tilde{\mathfrak{P}}_h(w)\to 1$ as $w\to B$ implies $\tilde{\mathfrak{P}}_{hv}=0$ at the endpoints

$w = \overline{W_0}, W_0$. Any even double antiderivative of $\wp$ equalling zero at $w = \overline{W_0}, W_0$ must be a constant times the function (13). For $\tilde{\mathfrak{P}}_{hv}$, the constant is set by

$$\tilde{\mathfrak{P}}'_{hv}(B) = \tilde{\mathfrak{P}}'_h(B) = 1/(C-B) = 1/(W_0 - \overline{W_0}) = \sqrt{3}/2i\omega_2, \tag{18}$$

together with the fact that $[\operatorname{Log}\sigma]'(B) = \zeta(\overline{W_0})$, the value of which is given above. By examination, the constant should be $3\sqrt{3}/\pi$; which yields (15). ∎

Numerical Remark. A power series expansion of $\wp(w)$ about $w = W_0$ that is accurate to $O((w-W_0)^{15})$ is given in ref. 1, Eq. (18.13.41). The corresponding expansion about $w = \overline{W_0}$, i.e., about B, is obtained by complex conjugation. By twice anti-differentiating this, an expansion of $\tilde{\mathfrak{P}}_{hv}(w)$ about $w = B$ accurate to $O((w-B)^{17})$ can be obtained.

Corollary 4.2.1. $\Pi_{hv}(1)$, the probability that all four sides of a large square are connected by a critical percolation cluster, equals

$$1/4 + (\sqrt{3}/4\pi)(3\log 3 - 4\log 2) \approx 0.322. \tag{19}$$

Proof. $\Pi_{hv}(r=1) = \mathfrak{P}_{hv}(z=1/2) = \tilde{\mathfrak{P}}_{hv}(w=\omega_2)$ by conformal invariance. This quantity can be computed from (15), using the closed-form expression for $\sigma(\omega_2)$ given at the beginning of this section. ∎

Alternative Proof. The expression (19) for $\Pi_{hv}(1)$ can be derived directly from Watts's formula, though the derivation is intricate; the following explains how. $\Pi_{hv}(1)$ equals $\mathfrak{P}_{hv}(1/2)$, i.e., $\mathfrak{P}_h(1/2) - \mathfrak{P}_{h\bar{v}}(1/2)$. By Formula 2.2,

$$\Pi_{hv}(1) = 1/2 - \frac{\sqrt{3}}{4\pi}\, {}_3F_2\left(\begin{matrix}1, 1, 4/3\\ 2, 5/3\end{matrix}\,\middle|\, 1/2\right), \tag{20}$$

since $\mathfrak{P}_h(1/2) = 1/2$. Summing the ${}_3F_2$ series requires care, since in general, it is harder to evaluate ${}_3F_2$ than ${}_2F_1$. For example, though Gauss's formula

$${}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}\,\middle|\, 1\right) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}, \qquad \Re(c-a-b) > 0, \tag{21}$$

evaluates any ${}_2F_1$ at unit argument, no general formula for ${}_3F_2(a, b, c; d, e; 1)$ in terms of gamma functions exists.[26, 27] However, certain special ${}_3F_2$'s can be evaluated at unit argument in closed form. At other argument values, the situation is unresolved. Many "strange" evaluations of ${}_3F_2$ and ${}_2F_1$ at rational points other than unity are known,[10] but most can apparently be reduced to evaluations at unity via appropriate transformations of the independent variable.

To move the ${}_3F_2$ evaluation point in (20) from $1/2$ to unity, the new quadratic transformation formula of Proposition 3.3 can be used. It yields

$$\Pi_{hv}(1) = 1/2 - \frac{\sqrt{3}}{8\pi}\, {}_3F_2\left(\begin{matrix}1, 1, 7/6\\ 2, 5/3\end{matrix}\,\middle|\, 1\right). \tag{22}$$

(The point $1/2$ is on the boundary of the region to which Proposition 3.3 applies; but by the $\Re(\sum \beta_i - \sum \alpha_i) > 0$ convergence criterion mentioned in the appendix, the equality of the proposition extends to the boundary.) Fortunately, the ${}_3F_2(1)$ in (22) can be evaluated in closed form. An ingenious application of L'Hôpital's rule to Gauss's formula shows that

$${}_3F_2\left(\begin{matrix}1, 1, a\\ 2, c\end{matrix}\,\middle|\, 1\right) = \frac{c-1}{a-1}\,[\psi(c-1) - \psi(c-a)], \qquad a \neq 1, \quad \Re(c-a) > 0 \tag{23}$$

(see Luke,[15] Section 5.2.4). Here $\psi \stackrel{\text{def}}{=} \Gamma'/\Gamma$ is the digamma function. So

$$\Pi_{hv}(1) = 1/2 - \frac{\sqrt{3}}{2\pi}\,[\psi(2/3) - \psi(1/2)]. \tag{24}$$

The values $\psi(2/3)$, $\psi(1/2)$ are $-\gamma + \pi/2\sqrt{3} - (3/2)\log 3$ and $-\gamma - 2\log 2$ respectively, where γ is Euler's constant. (See ref. 17, Vol. 2, App. II.3.) Substitution yields the expression (19) for $\Pi_{hv}(1)$. ▮

Proposition 4.3 (Cardy's New Formula, Transformed). If conformal invariance holds, Formula 2.3 corresponds on the equilateral triangle $\triangle ABC$ plus boundary to the following. $\tilde{\mathfrak{N}}_h(w)$, the expected number of percolation clusters connecting the boundary segments Bw and CA, is the restriction to BC of an analytic function. Explicitly, $\tilde{\mathfrak{N}}_h(w)$ equals

$$-\frac{\sqrt{3}}{4\pi}\left\{6\,\mathrm{Log}\,\sigma(w) + \mathrm{Log}\left[\frac{1}{2} - \frac{\wp'(w)}{2i}\right]\right\} + \frac{(3-\sqrt{3}\,i)}{4}\frac{w}{\omega_2} + \frac{\sqrt{3}\,i}{4}.$$

Proof. This follows from Corollary 2.3.1 by replacing $\mathfrak{N}_h(z)$, $\mathfrak{P}_h(z)$, $\mathfrak{P}_{hv}(z)$, z by $\tilde{\mathfrak{N}}_h(w)$, $\tilde{\mathfrak{P}}_h(w)$, $\tilde{\mathfrak{P}}_{hv}(w)$, $S(w)$, respectively, and substituting the expressions for $\tilde{\mathfrak{P}}_h(w)$, $\tilde{\mathfrak{P}}_{hv}(w)$ provided by Propositions 4.1 and 4.2. ▮

Corollary 4.3.1. $N_h(1)$, the expected number of critical percolation clusters crossing between opposite sides of a large square, equals

$$3/8 + (\sqrt{3}/8\pi)(3\log 3 - 2\log 2) \approx 0.507. \tag{25}$$

Proof. $N_h(r=1) = \mathfrak{N}_h(z=1/2) = \tilde{\mathfrak{N}}_h(w=\omega_2)$ by conformal invariance. This quantity can be computed from the formula for $\tilde{\mathfrak{N}}_h(w)$, using the known value of $\sigma(\omega_2)$ and the fact that $\wp'(\omega_2)=0$. More simply, it follows from Corollary 2.1 by substituting the expression for $\Pi_{hv}(r=1) = \mathfrak{P}_{hv}(z=1/2)$ provided by Corollary 4.2.1. ∎

Finally we come to Schramm's formula, Formula 2.4. It differs from Formulas 2.1–2.3 in that its restatement employs a triangular domain that is not equilateral. Let $\triangle A'B'C' \subset \mathbb{C}$ be an isosceles triangle with interior angles $2\pi/3$, $\pi/6$, $\pi/6$ at A', B', C', respectively. (For concreteness, take the vertices A', B', C' equal to $1+i\sqrt{3}/3$, 0, 2, respectively.) Special boundary conditions are imposed: the edge $B'C'$ is divided into $B'w$ and wC', and on the underlying discrete lattice, percolation along $B'w$ is allowed by fiat. Also, for percolation purposes the edges $A'B'$ and $A'C'$ are identified, so that in effect, the boundary of $\triangle A'B'C'$ comprises only the edge $B'C'$, and the vertex A' is in its interior.

Proposition 4.4 (Schramm's Formula, Transformed). If conformal invariance holds, Formula 2.4 corresponds on the triangle $\triangle A'B'C'$ plus boundary, with edges $A'B'$, $A'C'$ identified, to the following. Let $\tilde{\mathfrak{P}}_{\text{surr}}(w)$ denote the probability that the vertex A' is surrounded by the percolation hull of the boundary segment $B'w$, i.e., the outermost boundary of the percolation cluster growing from $B'w$. Then $\tilde{\mathfrak{P}}_{\text{surr}}(w)$ is the restriction to $B'C'$ of an analytic function that is *linear*. Explicitly, $\tilde{\mathfrak{P}}_{\text{surr}}(w) = (w-B')/(C'-B')$.

Proof. The first thing to observe is that up to trivial changes of the independent and dependent variables, the function $\mathfrak{P}_{\text{surr}}(z)$ of Formula 2.4 is identical to the function $\mathfrak{P}_h(z)$ of Cardy's Formula 2.1, or more accurately to its analytic continuation. This is the source of the linear behavior on $\triangle A'B'C'$. To see the close relation between the two functions, use Lemma 3.1 to pull back the P-symbol of Formula 2.4 via the quadratic map $z \mapsto -z^2$ on $\mathbb{CP}^1$. The result of this procedure is that $\mathfrak{P}_{\text{surr}}(z)$ equals $1/2$ plus a function in the solution space specified by

$$P\left\{\begin{array}{ccc|c} 0 & 1 & \infty & -z^2 \\ \hline 0 & 0 & 0 & \\ 1/2 & 1/3 & 1/6 & \end{array}\right\} = P\left\{\begin{array}{ccc|c} -i & i & \infty & z \\ \hline 0 & 0 & 0 & \\ 1/3 & 1/3 & 1/3 & \end{array}\right\}, \tag{26}$$

since the quadratic map has $0, \infty$ as its critical points (of multiplicity 2), and takes $0, \pm i, \infty$ to $0, 1, \infty$. There is no fourth column associated to

$z = 0$ in the right-hand P-symbol, since in the pulled-back ODE, $z = 0$ has exponents 0, 1 and is effectively an ordinary point. The close connection between this P-symbol and the P-symbol of Formula 2.1 is obvious. A careful computation, omitted here, yields

$$\mathfrak{P}_{\text{surr}}(z) = 1/2 + \text{const} \times [\mathfrak{P}_h(1/2 + iz/2) - 1/2]. \tag{27}$$

But for the purpose of proving the proposition, (26) will suffice.

Via a conformal map R similar to the map S used in the proofs of Propositions 4.1–4.3, the right-hand P-symbol in (26) can be pulled back to a trivial P-symbol. However, it will turn out that R maps a triangle $\triangle A'B'C'$ of the above form not onto $\mathbb{H}$, but rather onto the *slit* half plane $\mathbb{H} \setminus [i, +\infty i)$. The edges $A'B'$, $A'C'$ will be mapped to opposite sides of the slit, and will therefore need to be identified for percolation purposes. A' will be mapped to i, so the statements of the proposition and Formula 2.4 will correspond. The map R and $\triangle A'B'C'$ are chosen as follows.

The function $S(w) = 1/2 + \wp'(w)/2i$ maps $\triangle ABC = \triangle 0, \overline{W_0}, W_0$ onto $\mathbb{H}$, and its vertices to $\infty, 0, 1 \in \partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$. So $R(w) \stackrel{\text{def}}{=} \wp'(w)$ maps $\triangle 0, \overline{W_0}, W_0$ onto the right half plane, and its vertices to $\infty i, -i, i \in \mathbb{R}i \cup \{\infty i\}$. By reflecting through the line passing through $\overline{W_0}$ and W_0 (and their midpoint, the real half-period ω_2), it follows that as well, R maps the triangle $\triangle 2\omega_2, W_0, \overline{W_0}$ onto the left half plane. Therefore R maps the parallelogram without boundary $0, \overline{W_0}, 2\omega_2, W_0$ comprising these two equilateral triangles and the line segment $\overline{W_0}W_0$ (their common boundary) onto the doubly slit plane $\mathbb{C} \setminus [i, +\infty i) \setminus (-\infty i, -i]$. In particular, it maps the upper half of this parallelogram, the isosceles triangle $\triangle W_0, 0, 2\omega_2$, onto the slit upper half plane. This triangle has interior angles $2\pi/3, \pi/6, \pi/6$. Its vertices are mapped to $i, +\infty i, +\infty i$, respectively.

One accordingly chooses $\triangle A'B'C' = \triangle W_0, 0, 2\omega_2 \propto \triangle(1 + i\sqrt{3}/3), 0, 2$, where the constant of proportionality equals ω_2. The pullback proceeds as in the proof of Proposition 4.1. The right-hand P-symbol of (26) is pulled back via $z = R(w)$ to

$$P\left\{\begin{array}{ccc|c} [A] & [B] & [C] & w \\ \hline 0 & 0 & 0 & \\ 1 & 1 & 1 & \end{array}\right\}, \tag{28}$$

So the pulled-back ODE on $\triangle A'B'C'$ (or more generally, on $\mathbb{C}$) has no singular points and should be effectively $(d^2/dw^2)\, \tilde{\mathfrak{P}}_{\text{surr}}(w) = 0$, as can be verified by an explicit computation. Therefore $\tilde{\mathfrak{P}}_{\text{surr}}$ must be linear. Since $\tilde{\mathfrak{P}}_{\text{surr}}(B')=0$ and $\tilde{\mathfrak{P}}_{\text{surr}}(C')=1$, the proposition follows. ∎

APPENDIX A: HYPERGEOMETRIC FUNCTIONS, *P*-SYMBOLS

The following are facts about the generalized hypergeometric function ${}_{q+1}F_q$ and its ODE.[21] Let the rising factorial $\alpha(\alpha+1)\cdots(\alpha+k-1)$ be denoted $(\alpha)_k$; by convention, $(\alpha)_0$ is interpreted as unity. Then for any $q \geqslant 1$, the function ${}_{q+1}F_q(\alpha_1,..., \alpha_{q+1}; \beta_1,..., \beta_q; z)$ is defined by

$$ {}_{q+1}F_q\left(\begin{matrix}\alpha_1,..., \alpha_{q+1}\\ \beta_1,..., \beta_q\end{matrix}\,\middle|\, z\right) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_{q+1})_k}{(\beta_1)_k \cdots (\beta_q)_k} \frac{z^k}{k!}. \tag{A.1} $$

Provided no denominator parameter β_i is a nonpositive integer, the series coefficients are finite and the series converges absolutely on the open unit disk $|z|<1$. Provided $\Re(\sum \beta_i - \sum \alpha_i) > 0$, it converges on $|z|=1$, as well. Let $\vartheta \stackrel{\text{def}}{=} z\, d/dz$. Then on the disk, ${}_{q+1}F_q$ satisfies the order-$(q+1)$ ODE

$$ [\vartheta(\vartheta+\beta_1-1)\cdots(\vartheta+\beta_q-1) - z(\vartheta+\alpha_1)\cdots(\vartheta+\alpha_{q+1})]\, F = 0. \tag{A.2} $$

It is the only solution analytic at $z=0$ and equalling unity there.

The natural domain of definition of (A2) is the Riemann sphere $\mathbb{CP}^1 \stackrel{\text{def}}{=} \mathbb{C} \cup \infty$. By examination, this ODE has $z=0, 1, \infty$ as its only singular points, and is *Fuchsian*: each singular point is regular. ${}_{q+1}F_q$ can be continued to a meromorphic function on $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$, which is generally multivalued. In fact, the solution space of (A.2) is a $(q+1)$-dimensional space of multivalued meromorphic functions. To avoid multivaluedness, $\mathbb{CP}^1$ is cut along $[1, \infty]$. By definition, ${}_{q+1}F_q$ is the continuation of the series from the disk to $\mathbb{CP}^1 \setminus [1, \infty]$.

The solution space of any order-n Fuchsian ODE on $\mathbb{CP}^1$ with three singular points is determined to a large extent by their locations and the n characteristic exponents associated to each.[16] Let $z_1, z_2, z_3 \in \mathbb{CP}^1$ denote the singular points, and $\rho_1^{(i)},..., \rho_n^{(i)} \in \mathbb{C}$ the exponents of $z=z_i$. If $z_i \neq \infty$, this generally means that for each $j \in \{1,..., n\}$, the equation has a solution asymptotic to $(z-z_i)^{\rho_j^{(i)}}$ as $z \to z_i$. (If $z_i=\infty$, then $(z-z_j)^{\rho_j^{(i)}}$ must be interpreted as $z^{-\rho_j^{(i)}}$. Also, if the difference between any pair of exponents of a singular point is an integer, the solution corresponding to the smaller one may include a logarithmic factor.) This definition of characteristic exponents extends immediately to ordinary points. The n exponents of any finite ordinary point $z \neq z_1, z_2, z_3$ are $0, 1,..., n-1$.

The Riemann P-symbol for such an ODE, or for its solution space, is

$$P\left\{\begin{array}{ccc|c} z_1 & z_2 & z_3 & z \\ \hline \rho_1^{(1)} & \rho_1^{(2)} & \rho_1^{(3)} & \\ \vdots & \vdots & \vdots & \\ \rho_n^{(1)} & \rho_n^{(2)} & \rho_n^{(3)} & \end{array}\right\}, \tag{A.3}$$

where the order of exponents in each column is not significant. This tableau facilitates symbolic manipulation. For example, multiplying the general solution by $(z-z_0)^c$ will add c to the exponents of $z=z_0$ and $-c$ to the exponents of $z=\infty$.

It is readily verified that the ODE (A2) has exponents $0, 1-\beta_1,\ldots, 1-\beta_q$ at $z=0$, exponents $0, 1, 2,\ldots, q-1, s$ at $z=1$, and exponents $\alpha_1,\ldots, \alpha_{q+1}$ at $z=\infty$, where $s \overset{\text{def}}{=} \sum \beta_i - \sum \alpha_i$. (This seems not to be well known; it is only partially explained in ref. 21.) In an *ad hoc* notation, we write

$${}_{q+1}F_q\left(\begin{array}{c} \alpha_1,\ldots,\alpha_{q+1} \\ \beta_1,\ldots,\beta_q \end{array}\middle| z\right) \propto P\left\{\begin{array}{ccc|c} 0 & 1 & \infty & z \\ \hline \boxed{0} & 0 & \alpha_1 & \\ 1-\beta_1 & 1 & \alpha_2 & \\ \vdots & \vdots & \vdots & \\ 1-\beta_{q-1} & q-1 & \alpha_q & \\ 1-\beta_q & s & \alpha_{q+1} & \end{array}\right\}, \tag{A.4}$$

with the box indicating that ${}_{q+1}F_q$ belongs to the zero exponent at $z=0$. The sum of the $3(q+1)$ exponents equals $\binom{q+1}{2}$, though this property is not specific to the hypergeometric ODE: it holds for any order-$(q+1)$ Fuchsian ODE on $\mathbb{CP}^1$ with three singular points. So there are only $3q+2$ independent exponent parameters.

Any order-$(q+1)$ Fuchsian ODE on $\mathbb{CP}^1$ with three singular points, or more accurately its solution space, is characterized by the $3q+2$ independent exponent parameters and $\binom{q}{2}$ additional "accessory" parameters, which together with the exponent parameters determine the global monodromy. (See Poole,[16] Section 20, for a normal form for the ODE from which the parameters may be extracted, with some effort.) The second-order (i.e., $q=1$) case is special in that there are no accessory parameters, and the solution space of the ODE is uniquely determined by its P-symbol. This is not the case when $q \geqslant 2$, i.e., when the ODE is of third or higher order.

If the $q+1$ exponents of one of the three singular points are $0, 1, \ldots, q-1, s$ for some s, up to an overall additive constant, we say the ODE is of *hypergeometric type*. Provided its $\binom{q}{2}$ accessory parameters take suitable values, the solutions of any ODE of hypergeometric type can be expressed in terms of hypergeometric functions, since it can be transformed to the hypergeometric ODE by redefining its independent and dependent variables so as to move its singular points to $0, 1, \infty$, and remove the additive constant.

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